

# Large order Reynolds expansions for the Navier-Stokes equations

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## Abstract

We consider the Cauchy problem for the incompressible homogeneous Navier-Stokes (NS) equations on a  $d$ -dimensional torus (typically, with  $d = 3$ ), in the  $C^\infty$  formulation described, e.g., in [25]. In the cited work and in [22] it was shown how to obtain quantitative estimates on the exact solution of the NS Cauchy problem via the *a posteriori* analysis of an approximate solution; such estimates concern the interval of existence of the exact solution and its distance from the approximate solution, evaluated in terms of Sobolev norms. In the present paper we consider an approximate solutions of the NS Cauchy problem having the form  $u^N(t) = \sum_{j=0}^N R^j u_j(t)$ , where  $R$  is the “mathematical” Reynolds number (the reciprocal of the kinematic viscosity) and the coefficients  $u_j(t)$  are determined stipulating that the NS equations be satisfied up to an error  $O(R^{N+1})$ . This subject was already treated in [24], where, as an application, the Reynolds expansion of order  $N = 5$  in dimension  $d = 3$  was considered for the initial datum of Behr-Nečas-Wu (BNW). In the present paper, these results are enriched regarding both the theoretical analysis and the applications. Concerning the theoretical aspect, we refine the approach of [24] using results from [25]; moreover, we show how to take into account the symmetries of the initial datum in building up the expansion. Concerning the applicative aspect we consider two more ( $d = 3$ ) initial data, namely, the vortices of Taylor-Green (TG) and Kida-Murakami (KM); the Reynolds expansions for the BNW, TG and KM data are performed symbolically via a Python program, attaining orders between  $N = 12$  and  $N = 20$ . Our *a posteriori* analysis indicates, amongst else, that the solution of the NS equations with anyone of the above three data is global if  $R$  is below an explicitly computed critical value. Admittedly, our critical Reynolds numbers are below the ones characterizing the turbulent regime; however these bounds have a sound theoretical support, are fully quantitative and improve previous results of global existence.

**Keywords:** Navier-Stokes equations, existence and regularity theory, theoretical approximation. **AMS 2000 Subject classifications:** 35Q30, 76D03, 76D05.

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# 1 Introduction and preliminaries

**Navier-Stokes (NS) equations; Reynolds number.** The NS equations for an incompressible homogeneous fluid with no external forces, periodic boundary conditions and initial datum  $u_*$  can be written as

$$\frac{\partial u}{\partial \mathbf{t}} = \nu \Delta u + \mathcal{P}(u, u) , \quad u(x, 0) = u_*(x) . \quad (1.1)$$

Here:  $\nu \in (0, +\infty)$  is the kinematic viscosity;  $u = u(x, \mathbf{t})$  is the divergence free velocity field; the space variables  $x = (x_s)_{s=1, \dots, d}$  belong to the torus  $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$ ;  $\Delta := \sum_{s=1}^d \partial_{ss}$  is the Laplacian. Furthermore,  $\mathcal{P}$  is the bilinear map defined as follows: for all sufficiently regular velocity fields  $v, w$  on  $\mathbf{T}^d$ ,

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w) \quad (1.2)$$

where  $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$  ( $r = 1, \dots, d$ ) and  $\mathfrak{L}$  is the Leray projection onto the space of divergence free vector fields. As in [24], let us define

$$t := \nu \mathbf{t} , \quad R := \frac{1}{\nu} ; \quad (1.3)$$

then Eq. (1.1) takes the form

$$\frac{\partial u}{\partial t} = \Delta u + R \mathcal{P}(u, u) , \quad u(x, 0) = u_*(x) , \quad (1.4)$$

to which we systematically refer in the sequel. In the present framework, it is natural to define the Reynolds number as

$$Re := \frac{V_* L_*}{\nu} = V_* L_* R , \quad (1.5)$$

where  $V_*$  is a characteristic velocity and  $L_*$  a characteristic length; later on we will give precise definitions for  $V_*$  and  $L_*$  as quadratic means related to the initial datum, which fit well to our Sobolev framework (see Eqs.(1.22)-(1.24)). In the sequel  $R$  and  $Re$  will be referred to as the “mathematical” and the “physical” Reynolds number, respectively.

**NS functional setting.** The functional setting proposed in [22] for Eq. (1.4) was mainly based on  $L^2$ -type Sobolev spaces of finite order; in [25] the attention passed to Sobolev spaces of infinite order, made of  $C^\infty$  functions. Both references are relevant for our present purposes, so it convenient to review a few issues from each one.

Let us start from the space  $D'(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{D}'$  of  $\mathbf{R}^d$ -valued distributions on  $\mathbf{T}^d$ . Each  $v \in \mathbb{D}'$  has a weakly convergent Fourier expansion  $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$  where  $e_k(x) := e^{ik \bullet x}$  and the coefficients  $v_k \in \mathbf{C}^d$  fulfil the relations  $\overline{v_k} = v_{-k}$  due to the

reality of  $v$ . The Laplacian and the associated semigroup act on the whole space  $\mathbb{D}'$  and possess the Fourier representations

$$(\Delta v)_k = -|k|^2 v_k, \quad (e^{t\Delta} v)_k = e^{-t|k|^2} v_k \quad (1.6)$$

( $v \in \mathbb{D}'$ ,  $t \in [0, +\infty)$ ,  $k \in \mathbf{Z}^d$ ). In the sequel we consider the spaces  $L^p(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^p$  for  $p \in [1, +\infty)$ ; we are mainly interested in the case  $p = 2$ . For any  $n \in \mathbf{R}$ , the  $n$ -th Sobolev space of divergence free, zero mean vector fields on  $\mathbf{T}^d$  is

$$\begin{aligned} \mathbb{H}_{\Sigma_0}^n(\mathbf{T}^d) &\equiv \mathbb{H}_{\Sigma_0}^n := \{v \in \mathbb{D}' \mid \operatorname{div} v = 0, \langle v \rangle = 0, \sqrt{-\Delta}^n v \in \mathbb{L}^2\} \\ &= \{v \in \mathbb{D}' \mid k \bullet v_k = 0 \text{ for all } k, v_0 = 0, \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k| < +\infty\} \end{aligned} \quad (1.7)$$

(in the above  $\langle v \rangle$  indicates the mean over  $\mathbf{T}^d$ , that equals  $v_0$ ). This is a Hilbert space with the inner product and the norm

$$\langle v | w \rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w \rangle_{L^2} = (2\pi)^d \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} \overline{v_k} \bullet w_k, \quad (1.8)$$

$$\|v\|_n := \sqrt{\langle v | v \rangle_n} \quad (1.9)$$

( $a \bullet b := \sum_{r=1}^d a_r b_r$  for all  $a, b \in \mathbf{C}^d$ ). We have  $\mathbb{H}_{\Sigma_0}^p \hookrightarrow \mathbb{H}_{\Sigma_0}^n$  if  $n, p \in \mathbf{R}$  and  $n \leq p$ , where  $\hookrightarrow$  indicates a continuous embedding.

We consider as well the infinite order Sobolev space

$$\mathbb{H}_{\Sigma_0}^\infty(\mathbf{T}^d) \equiv \mathbb{H}_{\Sigma_0}^\infty := \cap_{n \in \mathbf{R}} \mathbb{H}_{\Sigma_0}^n = \cap_{n \in \mathbf{N}} \mathbb{H}_{\Sigma_0}^n; \quad (1.10)$$

this is a Fréchet space with the locally convex topology induced by the family of norms  $(\|\cdot\|_n)_{n \in \mathbf{R}}$  or, equivalently, by the countable subfamily  $(\|\cdot\|_n)_{n \in \mathbf{N}}$ . For  $k \in \mathbf{N} \cup \{\infty\}$ , let us consider the space

$$\mathbb{C}_{\Sigma_0}^k(\mathbf{T}^d) \equiv \mathbb{C}_{\Sigma_0}^k := \{v \in C^k(\mathbf{T}^d, \mathbf{R}^d) \mid \operatorname{div} v = 0, \langle v \rangle = 0\}, \quad (1.11)$$

which is a Banach space for  $k < \infty$  and a Fréchet space for  $k = \infty$ , when equipped with the sup norms for all derivatives up to order  $k$ . Let  $h, k \in \mathbf{N}$ ,  $n \in \mathbf{R}$ ; then

$$\mathbb{C}_{\Sigma_0}^h \hookrightarrow \mathbb{H}_{\Sigma_0}^n \text{ if } h \geq n, \quad \mathbb{H}_{\Sigma_0}^n \hookrightarrow \mathbb{C}_{\Sigma_0}^k \text{ if } n > k + d/2, \quad (1.12)$$

where the second statement depends on the Sobolev Lemma; these facts imply

$$\mathbb{H}_{\Sigma_0}^\infty = \mathbb{C}_{\Sigma_0}^\infty \quad (1.13)$$

(which indicates the equality of the above vector spaces and of their Fréchet topologies).

Let us pass to the map  $\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w)$  of Eq. (1.2). The following facts are known:

- (i) the map  $\mathcal{P}$  is well defined and bilinear from  $\mathbb{H}_{\Sigma_0}^0 \times \mathbb{H}_{\Sigma_0}^1$  to  $\mathfrak{L}_0^1$  (the image under the Leray projection of the  $L^1$ , zero mean vector fields). In terms of Fourier coefficients,

$$\mathcal{P}(v, w)_k = -i\mathfrak{L}_k \sum_{h \in \mathbf{Z}^d} [v_h \bullet (k - h)] w_{k-h} , \quad (1.14)$$

where  $\mathfrak{L}_k : \mathbf{C}^d \rightarrow \mathbf{C}^d$  is the projection onto the orthogonal complement of  $k$ .

- (ii) For each real  $n > d/2$ ,  $\mathcal{P}$  sends continuously  $\mathbb{H}_{\Sigma_0}^n \times \mathbb{H}_{\Sigma_0}^{n+1}$  to  $\mathbb{H}_{\Sigma_0}^n$ ; so, there is a constant  $K_{nd} \equiv K_n$  such that

$$\|\mathcal{P}(v, w)\|_n \leq K_n \|v\|_n \|w\|_{n+1} \quad \text{for } v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1} . \quad (1.15)$$

By the arbitrariness of  $n$ , one infers that  $\mathcal{P}$  sends continuously  $\mathbb{H}_{\Sigma_0}^\infty \times \mathbb{H}_{\Sigma_0}^\infty$  to  $\mathbb{H}_{\Sigma_0}^\infty$ . As a generalization of (1.15), for all real  $p, n$  such that  $p \geq n > d/2$  there is a constant  $K_{pnd} \equiv K_{pn}$  such that

$$\|\mathcal{P}(v, w)\|_p \leq \frac{1}{2} K_{pn} (\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}) \quad (1.16)$$

$$\text{for } v \in \mathbb{H}_{\Sigma_0}^p, w \in \mathbb{H}_{\Sigma_0}^{p+1} .$$

- (iii) For each real  $n > d/2 + 1$ , there is a constant  $G_{nd} \equiv G_n$  such that

$$|\langle \mathcal{P}(v, w) | w \rangle_n| \leq G_n \|v\|_n \|w\|_n^2 \quad \text{for } v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1} ; \quad (1.17)$$

this is the famous Kato inequality, see [8]. More generally, for all real  $p, n$  such that  $p \geq n > d/2 + 1$  there is constant  $G_{pnd} \equiv G_{pn}$  such that

$$|\langle \mathcal{P}(v, w) | w \rangle_p| \leq \frac{1}{2} G_{pn} (\|v\|_p \|w\|_n + \|v\|_n \|w\|_p) \|w\|_p \quad (1.18)$$

$$\text{for } v \in \mathbb{H}_{\Sigma_0}^p, w \in \mathbb{H}_{\Sigma_0}^{p+1} .$$

Papers [21] [23] give explicit (but probably non optimal) expressions for the constants  $K_n, G_n$  in the inequalities (1.15) (1.17); in particular, these references show that one can take

$$K_3 = 0.323 , \quad G_3 = 0.438 \quad \text{if } d = 3 . \quad (1.19)$$

Eqs. (1.16) and (1.18) are “tame” refinements (in the Nash-Moser sense) of (1.15) and (1.17), respectively, to which they are reduced for  $p = n$  with  $K_n = K_{nn}$  and  $G_n = G_{nn}$ . Some relations very similar to these tame inequalities have been used in [1] [31] and, more recently, in [29]. Appendix A of [25], anticipating a more detailed analysis to be presented in [17], gives explicit formulas for  $K_{pn}$  and  $G_{pn}$  for arbitrary  $p, n$ .

**The NS Cauchy problem, in a  $C^\infty$  formulation.** Let us choose

$$R \in (0, +\infty) , \quad u_* \in \mathbb{H}_{\Sigma_0}^\infty ; \quad (1.20)$$

the corresponding NS Cauchy problem is:

$$\text{Find } u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty) \text{ such that } \frac{du}{dt} = \Delta u + R \mathcal{P}(u, u), \quad u(0) = u_* . \quad (1.21)$$

It is known that (1.21) has a unique maximal (i.e., not extendable) solution  $u$ , whose domain  $[0, T)$  depends in principle on  $R$  and  $u_*$ ; this gives by restriction any other solution. Some classical references related to this subject are [1] [7] [8] [9] [12] [13] [31] (some of these works consider mainly a finite Sobolev order, but there are standard arguments for passing to the infinite order case, reviewed e.g. in [25]).

**The physical Reynolds number in terms of Sobolev norms.** Let us return to Eq. (1.5), defining the physical Reynolds number  $Re$  in terms of some characteristic velocity  $V_*$  and length  $L_*$ . In this work we intend  $V_*$  to be the initial mean quadratic velocity:

$$V_* := \sqrt{\frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} |u_*|^2 dx} = \frac{1}{(2\pi)^{d/2}} \|u_*\|_{L^2} . \quad (1.22)$$

Moreover we define the characteristic length  $L_*$  as a quadratic mean of  $2\pi/|k|$  over the Fourier modes of  $u_*$ , in the following way:

$$L_* := \sqrt{\frac{\sum_{k \in \mathbf{Z}^d \setminus \{0\}} (2\pi/|k|)^2 |u_{*k}|^2}{\sum_{k \in \mathbf{Z}^d \setminus \{0\}} |u_{*k}|^2}} = 2\pi \frac{\|u_*\|_{-1}}{\|u_*\|_{L^2}} . \quad (1.23)$$

Thus

$$Re = V_* L_* R = \frac{\|u_*\|_{-1}}{(2\pi)^{d/2-1}} R ; \quad (1.24)$$

this will be our standard throughout the paper (note that, differently from (1.23), (1.24) makes sense for  $u_* = 0$  as well).

In the sequel, for better convenience, any numerical estimate involving the Reynolds number will be given for both parameters  $R$  and  $Re$ .

**The Reynolds expansion and its a posteriori analysis.** For any  $N \in \{0, 1, 2, \dots\}$ , one can build an approximate solution of the Cauchy problem (1.21) of the form

$$u^N(t) = \sum_{j=0}^N R^j u_j(t) \quad (1.25)$$

where the functions  $u_0, \dots, u_N : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^\infty$  are determined so that  $u^N(0) = u_*$  and  $du^N/dt - \Delta u^N - R \mathcal{P}(u^N, u^N) = O(R^{N+1})$ . A detailed analysis of this approximation has been presented in [24] in a slightly different framework, based on Sobolev spaces of finite order; in the next section this construction will be proposed in a version based on  $\mathbb{H}_{\Sigma_0}^\infty$ , and integrated with some results for initial data having nontrivial symmetries. In a few words:

- (i) One has a recursion rule to compute  $u_0, u_1, \dots$ .
- (ii) Once  $u^N$  has been determined, it is possible to set up for it an *a posteriori* analysis. The essential step in this direction requires to choose a real  $n > d/2 + 1$  and fix the attention on the Sobolev norms

$$\|u^N(t)\|_n, \quad \|u^N(t)\|_{n+1}, \quad (1.26)$$

$$\left\| \left( \frac{du^N}{dt} - \Delta u^N - R \mathcal{P}(u^N, u^N) \right)(t) \right\|_n \quad (1.27)$$

which measure the “growth” and the “differential error” of  $u^N$  at order  $n$  or  $n + 1$ ; a more refined analysis can be performed considering, in addition to the norms (1.26) (1.27), their analogues of any order  $p > n$ . It is important to remark that all the above norms can be explicitly computed (or bounded from above) using only the known functions  $u_0, \dots, u_N$ .

Now, one applies to  $u^N$  the general method of [22] (inspired by [5] [27]) and [25] to get estimates on the maximal solution  $u$  of the problem (1.21) via a posteriori analysis of any approximate solution. In this approach, using the norms (1.26) (1.27) or some functions of time which bind them from above, one writes down the so-called *control Cauchy problem*: this consists of a first order ODE for an unknown function  $\mathcal{R}_n : [0, T_c) \rightarrow \mathbf{R}$ , supplemented with the initial condition  $\mathcal{R}_n(0) = 0$  (see the forthcoming Proposition 2.3). Assume this problem to have a solution  $\mathcal{R}_n$ , with a suitable domain  $[0, T_c)$ ; then  $\mathcal{R}_n$  is nonnegative, the maximal solution  $u$  of (1.21) has a domain larger than  $[0, T_c)$ , and

$$\|u(t) - u^N(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c). \quad (1.28)$$

In particular,  $u$  is global if  $T_c = +\infty$ . The solution  $\mathcal{R}_n$  of the control Cauchy problem is typically found numerically, using any standard package for the integration of ODEs. After computing  $\mathcal{R}_n$ , for any  $p > n$  one can use the analogues of order  $p$  of the norms (1.26) (1.27) (or some upper bounds of them) to determine explicitly a function  $\mathcal{R}_p : [0, T_c) \rightarrow \mathbf{R}$  (again nonnegative) such that

$$\|u(t) - u^N(t)\|_p \leq \mathcal{R}_p(t) \quad \text{for } t \in [0, T_c). \quad (1.29)$$

**Plan of the paper and main results.** In Section 2, combining results from [24] and [25] we present the Reynolds expansion in  $\mathbb{H}_{\Sigma_0}^\infty$  and its a posteriori analysis via the control Cauchy problem. Moreover we show that the symmetries of the initial datum are inherited by each term  $u_j$  in the Reynolds expansion; this fact allows to reduce the effort in the recursive computation of the  $u_j$ 's.

In Section 3 we present the Reynolds expansions for the ( $d = 3$ ) initial data of Behr-Nečas-Wu (BNW), Taylor-Green (TG) and Kida-Murakami (KM); the expansions have been performed up to the orders  $N = 20$  (for BNW and TG) and  $N = 12$  (for KM) allowing, amongst else, to infer the global existence of the NS equations

for  $R \leq 0.51$ ,  $R \leq 2.8$  and  $R \leq 0.61$ , respectively; in terms of the physical Reynolds number (1.24), we have global existence if  $Re \leq 7.84$ ,  $Re \leq 5.07$  and  $Re \leq 1.00$ , respectively. In all these cases, the Reynolds expansions have been computed symbolically using Python programs written for this purpose; the orders  $N = 20$  or  $N = 12$  are the largest ones allowed by the PC we have used to run these programs (see Section 2 for more details). In the case of the BNW datum, the present computations improve the results obtained in [24] with an expansion up to order  $N = 5$ , computed symbolically via Mathematica (on this point, see also item (v) in the next paragraph).

In each one of the above three cases, the symmetries of the initial datum have been employed to reduce the computational costs. These symmetries are described in Appendix A; they are particularly relevant for the KM datum, that in fact arose in the papers by Kida and Murakami as a result of their investigation on the highly symmetric vector fields on  $\mathbf{T}^3$ .

Finally, in Section 4 we present some speculations on how to push to higher Reynolds numbers the present results of global existence for the BNW, TG and KM data.

**An assessment of the previous quantitative bounds.** We are aware that our upper bounds on  $Re$  for global existence, of order 10 at most, are much below the numerical values of  $Re$  related to turbulence: for example in the classical papers by Brachet *et al* [3] and Kida-Murakami [10] [11], where turbulence is analyzed numerically for the TG and KM initial data, the order of magnitude of  $Re$  is between  $10^2$  and  $10^4$ . However the cited works, and all the other investigations on turbulence of which we are aware, essentially assume global existence without proof.

We presume that our upper bounds on  $Re$ , being supported by a rigorous theoretical analysis, may have some interest; their small values correspond to the current state of the art concerning global existence of strong NS solutions and are, in any case, an improvement with respect to the estimates one could derive from previous quantitative approaches to the problem. For a better understanding of the last statement let us mention some results from earlier papers, and/or their implications in the case of the BNW datum:

- (i) For  $d = 3$ , paper [27] states global existence for the NS Cauchy problem in  $\mathbb{H}_{\Sigma_0}^1$  whenever  $R \leq 0.00724/\|u_*\|_1$  (see also the related works [6] [28]).
- (ii) Paper [20] improves the  $d = 3$  bound of [27] to  $R \leq 0.407/\|u_*\|_1$  (again, for any initial datum in  $\mathbb{H}_{\Sigma_0}^1$ ); for the BNW, TG and KM data this improved condition reads  $R \leq 0.00527$ ,  $R \leq 0.0298$ ,  $R \leq 0.00899$  respectively, or (using the definition (1.24))  $Re \leq 0.0811$ ,  $Re \leq 0.0541$ ,  $Re \leq 0.0147$ .
- (iii) Again for  $d = 3$ , papers [21][24] ensure global existence for the NS Cauchy problem (1.21) whenever  $R \leq 1/(G_3\|u_*\|_3) = 1/(0.438\|u_*\|_3)$ . In the BNW, TG and KM and cases, this general estimate reads  $R \leq 0.0147$ ,  $R \leq 0.0557$ ,  $R \leq 0.00458$ , respectively, or  $Re \leq 0.227$ ,  $Re \leq 0.101$ ,  $Re \leq 0.00752$ .
- (iv) Paper [22] uses a Galerkin approximant and its a posteriori analysis to infer global existence for the BNW datum under the condition  $R \leq 0.125$ , or  $Re \leq 1.92$ .
- (v) We have already recalled that [24] presents an  $N = 5$  Reynolds expansion for the BNW datum; this gives global existence when  $R \leq 0.23$ , or  $Re \leq 3.53$ .

The above results on the BNW, TG and KM cases are always weaker (or much weaker) than the present outcomes  $R \leq 0.51$ ,  $R \leq 2.8$ ,  $R \leq 0.61$  or  $Re \leq 7.84$ ,  $Re \leq 5.07$ ,  $Re \leq 1.00$ . As already mentioned, in Section 4 we will indicate some strategies that might yield future improvements.

For completeness, let us also mention papers [4] [14] [26] (and references therein); these present (quantitative or semiquantitative) conditions for global existence of strong NS solutions in three dimensions with periodic boundary conditions, for suitable initial data that, in our language, would correspond to large values of  $Re$ ; however these data have small periods (i.e., fast oscillations) in one space direction. On the contrary the BNW, TG and KM data considered in this work are not highly oscillating in any direction.

## 2 The Reynolds expansion: recursion rules, a posteriori analysis and symmetries

From now on  $K_n$ ,  $G_n$  are constants fulfilling the inequalities (1.15) (1.17);  $K_p$ ,  $G_p$  and  $K_{pn}$ ,  $G_{pn}$  are constants fulfilling Eqs. (1.15)(1.17) with  $n$  replaced by  $p$ , and Eqs. (1.16)(1.18).

**The expansion and its a posteriori analysis.** Let us consider the NS Cauchy problem (1.21) with  $R \in (0, +\infty)$  and a datum  $u_* \in \mathbb{H}_{\Sigma_0}^\infty$ . We choose an order  $N \in \{0, 1, 2, \dots\}$  and consider a function of the form

$$u^N : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^\infty, \quad t \mapsto u^N(t) := \sum_{j=0}^N R^j u_j(t), \quad (2.1)$$

$$u_j \in C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty) \quad \text{for } j = 0, \dots, N;$$

the functions  $u_j$  herein are to be determined. We regard  $u^N$  as an “approximate solution” of the NS Cauchy problem.

**2.1 Proposition.** (i) Let  $u^N$  be as in (2.1); then

$$\begin{aligned} & \frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) \\ &= \left( \frac{du_0}{dt} - \Delta u_0 \right) + \sum_{j=1}^N R^j \left[ \frac{du_j}{dt} - \Delta u_j - \sum_{\ell=0}^{j-1} \mathcal{P}(u_\ell, u_{j-\ell-1}) \right] - \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \mathcal{P}(u_\ell, u_{j-\ell-1}). \end{aligned} \quad (2.2)$$

(ii) One can define recursively a family of functions  $u_j \in C^\infty([0, +\infty), \mathbb{H}_{\Sigma_0}^\infty)$  prescribing the following, for  $t \in [0, +\infty)$ :

$$u_0(t) := e^{t\Delta} u_*, \quad (2.3)$$

$$u_j(t) := \sum_{\ell=0}^{j-1} \int_0^t ds e^{(t-s)\Delta} \mathcal{P}(u_\ell(s), u_{j-\ell-1}(s)) \quad (j = 1, \dots, N). \quad (2.4)$$



With this choice we have  $u_0(0) = u_*$ ,  $u_j(0) = 0$  for  $j = 1, \dots, N$  and the coefficients of  $R^0, R^1, \dots, R^N$  in the right hand side of Eq. (2.2) vanish, so that

$$u^N(0) = u_* , \quad (2.5)$$

$$\frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) = - \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \mathcal{P}(u_\ell, u_{j-\ell-1}) . \quad (2.6)$$

(iii) For any real  $n > d/2$ , Eqs. (2.6)(1.15) imply the following for  $t \in [0, +\infty)$ :

$$\begin{aligned} & \left\| \left( \frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) \right)(t) \right\|_n \\ & \leq K_n \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \|u_\ell(t)\|_n \|u_{j-\ell-1}(t)\|_{n+1} . \end{aligned} \quad (2.7)$$

More generally, for any real  $p \geq n > d/2$ , Eqs. (2.6)(1.16) imply

$$\begin{aligned} & \left\| \left( \frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) \right)(t) \right\|_p \\ & \leq \frac{1}{2} K_{pn} \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N (\|u_\ell(t)\|_p \|u_{j-\ell-1}(t)\|_{n+1} + \|u_\ell(t)\|_n \|u_{j-\ell-1}(t)\|_{p+1}) . \end{aligned} \quad (2.8)$$

**Proof.** An elementary variation of the proof of Proposition 3.1 in [24] (this considers the same subject for an initial datum  $u_*$  in a finite order Sobolev space, so that the functions  $u_j, u^N$  have less regularity; the adaptation to the present  $\mathbb{H}_{\Sigma_0}^\infty$  framework is straightforward).  $\square$

**2.2 Remarks.** (i) Eqs. (2.5) (2.6) indicate the following:  $u^N$  satisfies the initial condition of the NS Cauchy problem (1.21), and it fulfils the evolution equation in (1.21) up to an error described explicitly by (2.6).

(ii) The recursive computation of  $u_0, u_1, \dots, u_N$  via Eqs. (2.3) (2.4) can be performed in terms of Fourier coefficients; one uses the representations (1.6) and (1.14) for  $e^{t\Delta}$  and  $\mathcal{P}$ . Due to the structure of the recursion relations, the Fourier coefficients of  $u_0, u_1, \dots$  contain functions of time of the form  $B_{a,b}(t) := t^a e^{-bt}$  with  $a, b \in \mathbf{N}$ ; as already mentioned in [24], the related computations involve integrals of the form

$$\int_0^t ds e^{-|k|^2(t-s)} B_{a,b}(s) = \begin{cases} a! \left( \frac{B_{0,|k|^2}(t)}{(b - |k|^2)^{a+1}} - \sum_{\ell=0}^a \frac{B_{\ell,b}(t)}{(b - |k|^2)^{a+1-\ell} \ell!} \right) & \text{if } b \neq |k|^2; \\ \frac{B_{a+1,|k|^2}(t)}{a+1} & \text{if } b = |k|^2. \end{cases} \quad (2.9)$$

The calculation of  $u_0, u_1, \dots$  via the above rules is particularly simple if the initial datum  $u_*$  is a Fourier polynomial, i.e., if it has finitely many nonzero Fourier coefficients. In this case all the iterates  $u_0, \dots, u_N$  are Fourier polynomials as well, and

each one of their coefficients is a sum  $\sum_{a,b} C_{a,b} B_{a,b}(t)$  with  $(a, b)$  in a finite subset of  $\mathbf{N} \times \mathbf{N}$  and  $C_{a,b} \in \mathbf{C}^d$ .  $\square$

Keeping in mind the previous facts, one can treat  $u^N$  using the general framework of [22] [25] for approximate solutions of the NS equations; this analysis of the Reynolds expansion was performed in [24] at the level of finite order Sobolev spaces, and its analogue in  $\mathbb{H}_{\Sigma_0}^\infty$  is presented hereafter. Due to the strict connection between the present considerations and the inequalities (1.15)-(1.18), here and in the sequel we always use Sobolev norms of order  $> d/2 + 1$ .

**2.3 Proposition.** (i) With  $R, u_*$  as in (1.20) and  $N \in \{0, 1, 2, \dots\}$ , let  $u^N$  and  $u_0, \dots, u_N$  be as in Eqs. (2.1) (2.3) (2.4); moreover, choose a real  $n > d/2 + 1$ . Let  $\mathcal{D}_n, \mathcal{D}_{n+1}, \epsilon_n \in C([0, +\infty), [0, +\infty))$  be growth and error estimators for  $u^N$  of Sobolev orders  $n$  or  $n + 1$ , in the following sense:

$$\|u^N(t)\|_n \leq \mathcal{D}_n(t) , \quad \|u^N(t)\|_{n+1} \leq \mathcal{D}_{n+1}(t) , \quad (2.10)$$

$$\left\| \left( \frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) \right)(t) \right\|_n \leq \epsilon_n(t) \quad (2.11)$$

for all  $t \in [0, +\infty)$ . Moreover, assume there is a function  $\mathcal{R}_n \in C^1([0, T_c], \mathbf{R})$  solving the “control Cauchy problem”

$$\frac{d\mathcal{R}_n}{dt} = -\mathcal{R}_n + R(G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \mathcal{R}_n + R G_n \mathcal{R}_n^2 + \epsilon_n , \quad \mathcal{R}_n(0) = 0 . \quad (2.12)$$

If  $u \in C^\infty([0, T), \mathbb{H}_{\Sigma_0}^\infty)$  is the maximal solution of the NS Cauchy problem (1.21), one has

$$T \geq T_c , \quad (2.13)$$

$$\|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c) \quad (2.14)$$

In particular,  $u$  is global ( $T = +\infty$ ) if the control Cauchy problem (2.12) has a global solution ( $T_c = +\infty$ ).

(ii) Consider a real  $p > n$  and let  $\mathcal{D}_p, \mathcal{D}_{p+1}, \epsilon_p \in C([0, +\infty), [0, +\infty))$  be growth and error estimators of orders  $p$  or  $p + 1$ , fulfilling inequalities of the form (2.10) (2.11) with  $n$  replaced by  $p$ . Let  $\mathcal{R}_p \in C([0, T_c], \mathbf{R})$  be the solution of the linear problem

$$\frac{d\mathcal{R}_p}{dt} = -\mathcal{R}_p + R(G_p \mathcal{D}_p + K_p \mathcal{D}_{p+1} + G_{pn} \mathcal{R}_n) \mathcal{R}_p + \epsilon_p , \quad \mathcal{R}_p(0) = 0 , \quad (2.15)$$

which is given explicitly by

$$\mathcal{R}_p(t) = e^{-t} + R \mathcal{A}_p(t) \int_0^t ds e^s - R \mathcal{A}_p(s) \epsilon_p(s) \quad \text{for } t \in [0, T_c) , \quad (2.16)$$

$$\mathcal{A}_p(t) := \int_0^t ds (G_p \mathcal{D}_p(s) + K_p \mathcal{D}_{p+1}(s) + G_{pn} \mathcal{R}_n(s)) .$$

Then

$$\|u(t) - u_a(t)\|_p \leq \mathcal{R}_p(t) \quad \text{for } t \in [0, T_c) \quad (2.17)$$

(incidentally, note that (2.14) (2.17) imply  $\mathcal{R}_n(t), \mathcal{R}_p(t) \geq 0$ ).

**Proof.** Use items (i) (ii) in Proposition 4.4 of [25], choosing as an approximate solution for the NS Cauchy problem the function  $u^N$ ; note that the physical time of [25], say  $\mathfrak{t}$ , is related to the present time variable by  $t = \nu \mathfrak{t} = \mathfrak{t}/R$ . <sup>(2)</sup>  $\square$

**2.4 Remarks.** (i) In a few words, the previous proposition indicates how to obtain bounds on the interval of existence of  $u$  and on its distance from  $u^N$  of order  $n$ , or  $p > n$ , via an a posteriori analysis of  $u^N$ . Note that the estimators  $\mathcal{D}_n, \mathcal{D}_{n+1}, \epsilon_n$  appearing in the control Cauchy problem (2.12) can be constructed using only  $u^N$  (or its coefficients  $u_0, \dots, u_N$ ); the same can be said of the estimators  $\mathcal{D}_p, \mathcal{D}_{p+1}, \epsilon_p$  in (2.15).

(ii) The simplest choices for the above estimators are the tautological ones: if  $m$  is anyone of the Sobolev orders  $n, n+1, p, p+1$  mentioned before, take for  $\mathcal{D}_m$  and  $\epsilon_m$  the  $m$ -th norms of  $u^N = \sum_{j=0}^N R^j u_j$  and of  $du^N/dt - \Delta u^N - R\mathcal{P}(u^N, u^N)$ , as given by (2.6). One could consider alternative estimators, which are rougher but computable with a smaller effort; these have the form

$$\mathcal{D}_m(t) := \sum_{j=0}^N R^j \|u_j(t)\|_m \quad (2.18)$$

$$\epsilon_m(t) := K_m \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \|u_\ell(t)\|_m \|u_{j-\ell-1}(t)\|_{m+1} . \quad (2.19)$$

The prescription (2.19) is suggested by Eq. (2.7); an obvious variation of it for  $m = p$  is suggested by Eq. (2.8). The choices (2.18) (2.19) reduce the construction of the estimators to calculating the norms  $\|u_j(t)\|_m$  or  $\|u_j(t)\|_{m+1}$ , which is less expensive than computing exactly the norms of  $u^N$  and  $du^N/dt - \Delta u^N - R\mathcal{P}(u^N, u^N)$ . An intermediate alternative is to compute exactly the involved norms up to some order  $M \in \{1, \dots, N\}$  in  $R$ , and bind the reminders more roughly; this yields, for example, the estimators

$$\mathcal{D}_m(t) := \left\| \sum_{j=0}^M R^j u_j(t) \right\|_m + \sum_{j=M+1}^N R^j \|u_j(t)\|_m , \quad (2.20)$$

that will appear in the applications of the next section.

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<sup>2</sup>The cited reference indicates that the conditions for  $\mathcal{R}_n, \mathcal{R}_p$  in (2.12) (2.15) could be generalized replacing everywhere the equality sign  $=$  with  $\geq$ ; these generalizations are not relevant for our present purposes.

(iii) Eqs. (2.14)(2.17) entail some rather obvious bounds on the difference between the Fourier coefficients of  $u(t)$  and  $u^N(t)$ ; for example, as mentioned in [24], Eq. (2.14) implies

$$(2\pi)^{d/2} |u_k(t) - u_k^N(t)| \leq \frac{\mathcal{R}_n(t)}{|k|^n} \quad \text{for } k \in \mathbf{Z}^d \setminus \{0\} \text{ and } t \in [0, T_c) . \quad (2.21)$$

**Using the symmetries of the initial datum.** The present paragraph is inspired by a setting proposed in [15] to treat symmetries of the incompressible Euler equations, that we are presently adapting to the NS case (in any space dimension  $d$ ). We consider the group  $O(d, \mathbf{Z})$ , formed by the orthogonal  $d \times d$  matrices with integer entries:

$$O(d, \mathbf{Z}) := \{S \in \text{Mat}(d \times d, \mathbf{Z}) \mid S^T S = \mathbf{1}_d\} . \quad (2.22)$$

A  $d \times d$  matrix  $S$  belongs to  $O(d, \mathbf{Z})$  if and only if

$$S = \text{diag}(\epsilon_1, \dots, \epsilon_d) Q(\sigma) \quad (2.23)$$

$\epsilon_s \in \{\pm 1\}$ ,  $Q(\sigma)$  the matrix of a permutation  $\sigma : \{1, \dots, d\} \rightarrow \{1, \dots, d\}$ ;

more precisely,  $Q(\sigma)$  is the matrix such that  $(Q(\sigma)c)_s = c_{\sigma(s)}$  for all  $c \in \mathbf{C}^d$ ,  $s \in \{1, \dots, d\}$ . Incidentally, the representation (2.23) implies that each element of  $S$  takes values in  $\{-1, 0, 1\}$ . Counting the choices for the signs  $\epsilon_s$  and for  $\sigma$  in Eq. (2.23), one concludes that  $O(d, \mathbf{Z})$  has  $2^d \times d!$  elements. In particular,  $O(3, \mathbf{Z})$  has 48 elements; this group is often indicated with  $O_h$ , and referred to as the *octahedral group*.

To go on let us consider the semidirect product  $O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$ , i.e., the Cartesian product  $O(d, \mathbf{Z}) \times \mathbf{T}^d$ , viewed as a group with the composition law

$$(S, a)(U, b) := (SU, a + Sb) \quad (S, U \in O(d, \mathbf{Z}) ; a, b \in \mathbf{T}^d) . \quad (2.24)$$

Each element  $(S, a) \in O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$  induces a rototranslation

$$\mathcal{E}(S, a) : \mathbf{T}^d \rightarrow \mathbf{T}^d , \quad x \mapsto \mathcal{E}(S, a)(x) := Sx + a , \quad (2.25)$$

and the mapping  $(S, a) \mapsto \mathcal{E}(S, a)$  is a group homomorphism between  $O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$  and the group of diffeomorphisms of  $\mathbf{T}^d$  into itself.

Given a vector field  $v \in \mathbb{H}_{\Sigma_0}^\infty$  and  $(S, a) \in O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$ , we can construct the push-forward of  $v$  along the mapping  $\mathcal{E}(S, a)$ ; this is a vector field  $\mathcal{E}_*(S, a)v \in \mathbb{H}_{\Sigma_0}^\infty$  given by

$$\mathcal{E}_*(S, a)v : \mathbf{T}^d \rightarrow \mathbf{R}^d , \quad x \mapsto (\mathcal{E}_*(S, a)v)(x) = Sv(S^T(x - a)) , \quad (2.26)$$

and its Fourier coefficients are

$$(\mathcal{E}_*(S, a)v)_k = e^{-ia \bullet k} S v_{S^T k} \quad (k \in \mathbf{Z}^d) . \quad (2.27)$$

The linear map  $\mathcal{E}_*(S, a) : v \mapsto \mathcal{E}_*(S, a)v$  preserves the inner product  $\langle \mid \rangle_n$  for each real  $n$ ; denoting with  $O(\mathbb{H}_{\Sigma_0}^\infty)$  the group of linear operators of  $\mathbb{H}_{\Sigma_0}^\infty$  into itself preserving all inner products  $\langle \mid \rangle_n$ , we have an injective group homomorphism

$$\mathcal{E}_* : O(d, \mathbf{Z}) \ltimes \mathbf{T}^d \rightarrow O(\mathbb{H}_{\Sigma_0}^\infty) , \quad (S, a) \mapsto \mathcal{E}_*(S, a) . \quad (2.28)$$

Let us turn the attention to the NS equations. Using (2.27) with the Fourier representations (1.6) for  $\Delta$ ,  $e^{t\Delta}$  and (1.14) for  $\mathcal{P}$ , one infers

$$\Delta \mathcal{E}_*(S, a) v = \mathcal{E}_*(S, a) \Delta v, \quad e^{t\Delta} \mathcal{E}_*(S, a) v = \mathcal{E}_*(S, a) e^{t\Delta} v, \quad (2.29)$$

$$\mathcal{P}(\mathcal{E}_*(S, a) v, \mathcal{E}_*(S, a) w) = \mathcal{E}_*(S, a) \mathcal{P}(v, w) \quad (2.30)$$

for  $v, w \in \mathbb{H}_{\Sigma_0}^\infty$ . Now, consider the initial datum  $u_*$  for the NS Cauchy problem (1.21) and define the following, for  $\sigma \in \{+, -\}$ :

$$\mathcal{H}^\sigma(u_*) := \{(S, a) \in O(d, \mathbf{Z}) \ltimes \mathbf{T}^d \mid \mathcal{E}_*(S, a) u_* = \sigma u_*\}, \quad (2.31)$$

$$\mathcal{H}_r^\sigma(u_*) := \{S \in O(d, \mathbf{Z}) \mid (S, a) \in \mathcal{H}^\sigma(u_*) \text{ for some } a \in \mathbf{T}^d\}. \quad (2.32)$$

For  $\sigma = +$ ,  $\mathcal{H}^\sigma(u_*)$  and  $\mathcal{H}_r^\sigma(u_*)$  are subgroups of  $O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$  and  $O(d, \mathbf{Z})$ ; they will be called the *symmetry group* and the *reduced symmetry group* of  $u_*$ . For  $\sigma = -$ ,  $\mathcal{H}^\sigma(u_*)$  and  $\mathcal{H}_r^\sigma(u_*)$  will be called the *pseudo-symmetry* and *reduced pseudo-symmetry spaces* of  $u_*$ . The unions  $\mathcal{H}^+(u_*) \cup \mathcal{H}^-(u_*)$  and  $\mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)$  are subgroups of  $O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$  and  $O(d, \mathbf{Z})$ , respectively. If  $(\bar{S}, \bar{a})$  is any element of  $\mathcal{H}^-(u_*)$ , then  $\mathcal{H}^-(u_*) = \mathcal{H}^+(u_*) \circ (\bar{S}, \bar{a}) = (\bar{S}, \bar{a}) \circ \mathcal{H}^+(u_*)$  and  $\mathcal{H}_r^-(u_*) = \mathcal{H}_r^+(\bar{S}) = \bar{S} \mathcal{H}_r^+(u_*)$  (here  $\mathcal{H}^+(u_*) \circ (\bar{S}, \bar{a})$  means  $\{(S, a) \circ (\bar{S}, \bar{a}) \mid (S, a) \in \mathcal{H}^+(u_*)\}$ , and so on).

Using Eqs. (2.29) (2.30), one readily finds that the iterates defined by (2.3) (2.4) fulfil at all times the relations

$$\mathcal{E}_*(S, a) u_j(t) = \sigma^{j+1} u_j(t) \quad \text{for } (S, a) \in \mathcal{H}^\sigma(u_*), j \in \{0, 1, \dots, N\}. \quad (2.33)$$

Recalling Eq. (2.27), one can rephrase the above result in terms of Fourier coefficients, in the following way:

$$u_{j, Sk}(t) = \sigma^{j+1} e^{-ia \bullet Sk} S u_{j, k}(t) \quad \text{for } (S, a) \in \mathcal{H}^\sigma(u_*), j \in \{0, 1, \dots, N\}, k \in \mathbf{Z}^d. \quad (2.34)$$

Let us point out the implications of the above results in a concrete application of the recursion scheme (2.3) (2.4), say, for  $d = 3$ , and with a Fourier polynomial as an initial datum; in this case, Eq. (2.34) can be employed to reduce the computational cost for any iterate  $u_j$ . In fact, after computing a Fourier coefficient  $u_{j, k}$  one immediately obtains from (2.34) the coefficients  $u_{j, Sk}$  for all  $S$  in  $\mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)$ : it suffices to apply the cited equation, choosing for  $a$  any element of  $\mathbf{Z}^d$  such that  $(S, a) \in \mathcal{H}^\pm(u_*)$ . Note that  $\{Sk \mid S \in \mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)\}$  is the orbit of  $k$  with respect to the action of the group  $\mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)$  on  $\mathbf{Z}^3$ .

### 3 Applications. The BNW, TG and KM initial data

From here to the end of the paper, we consider the NS Cauchy problem (1.21) with

$$d = 3 ; \quad (3.1)$$

for the moment, the initial datum  $u_* \in \mathbb{H}_{\Sigma_0}^\infty$  is unspecified.

We are interested in the Reynolds expansion  $u^N(t) = \sum_{j=0}^N R^j u_j(t)$ , for suitable  $N$ , and on its a posteriori analysis via the control Cauchy problem (2.12). This will be performed choosing the Sobolev order

$$n = 3 ; \quad (3.2)$$

thus Eq. (2.12) takes the form

$$\frac{d\mathcal{R}_3}{dt} = -\mathcal{R}_3 + R(G_3\mathcal{D}_3 + K_3\mathcal{D}_4)\mathcal{R}_3 + R G_3\mathcal{R}_3^2 + \epsilon_3 , \quad \mathcal{R}_3(0) = 0 , \quad (3.3)$$

with  $K_3$  and  $G_3$  as in (1.19); in the sequel we denote with

$$\mathcal{R}_3 \in C^1([0, T_c], \mathbf{R}) \quad (3.4)$$

the maximal solution, which is nonnegative.

Let us recall that, after solving Eq. (3.3), for any real  $p > 3$  we could build a function  $\mathcal{R}_p \in C^1([0, T_c], \mathbf{R})$  following Eq. (2.16), to be used in relation to estimates of Sobolev order  $p$ ; the actual computation of these higher order bounds, in a number of applications, will be presented elsewhere.

**Choice of  $u_*$ ; automatic computations.** As anticipated in the Introduction, in this paper we consider the BNW, TG and KM initial data; these are Fourier polynomials, described in detail in the sequel.

For each one of these three data, the terms  $u_0, u_1, \dots$ , in the Reynolds expansion have been computed symbolically using Python on a PC. To this purpose, we have developed the following software utilities:

- (a) First of all, we have written a Python program working in principle for any initial datum  $u_*$  of polynomial type; this computes  $u_0, u_1, \dots$  using Eqs. (2.3) (2.4) and (1.6) (1.14).
- (b) Secondly, for each one the BNW, TG and KM data we have devised an *ad hoc* variant of the basic program in (a), implementing the symmetries of the datum (see Eq. (2.34) and the related comments). These variants reduce the computational costs, thus allowing to push the Reynolds expansion to higher orders than the ones allowed by the program in (a).

All the above Python programs use the package GMPY [32] for fast arithmetics on rational numbers; they have been run on an 8 Gb RAM PC. Using the program mentioned in (a) we have attained the orders  $N = 16, 14, 7$  for the BNW, TG and KM datum, respectively. Next we have used the specific Python programs mentioned in (b), implementing the symmetries of these data; this has allowed us to reach the orders  $N = 20, 20, 12$ , respectively <sup>(3)</sup>. The above Python programs also give analytic expressions for the estimators  $\mathcal{D}_3, \mathcal{D}_4$  and  $\epsilon_3$  appearing in the control equation (3.3). For the three data mentioned before we have used the estimator  $\epsilon_3$  defined via (2.19), and the estimators  $\mathcal{D}_3, \mathcal{D}_4$  defined via (2.20), with  $M = 5$ .

The KM case has required the longest computational times; calculations up to the order  $N = 12$  have taken, approximately, 90 hours for the determination of the  $u_j$ 's and 30 hours to compute the norms in Eqs. (2.19) and (2.20). Computations up to  $N = 20$  for the BNW and TG data have been a bit faster, but in any case have required a few days.

After computing the Reynolds expansion and the related estimators, one can solve numerically the control Cauchy problem (3.3). This involves a Riccati type ODE with time dependent coefficients, which have very long analytic expressions when the order  $N$  of the expansion is large; for the numerical treatment of this ODE we have used Mathematica on a PC. In our initial attempts, some numerical instabilities have appeared for large  $N$  in the integration of (3.3); these were due to insufficient precision in the numerical evaluation of  $\mathcal{D}_3, \mathcal{D}_4$  and  $\epsilon_3$  at the discrete times prescribed by the Mathematica routines for ODEs. To eliminate such instabilities, for the numerical integration of (3.3) we have replaced  $\mathcal{D}_3, \mathcal{D}_4$  and  $\epsilon_3$  with convenient interpolants, built by the internal routines of Mathematica after high precision computations of the norms in Eqs. (2.19) (2.20) at suitable grids of values for  $t$ . For the choices of  $N$  and  $R$  considered in our computations, the high precision computations of the norms at a grid of instants and the construction of the interpolants has required half an hour at most; after this, the numerical solution of (3.3) has been almost instantaneous. <sup>(4)</sup>

**General structure of the results from the control Cauchy problem.** Let  $u_*$  be the BNW, TG or KM datum. For any order  $N$  considered in our computations, the numerical solution of the control problem (3.3) for several values of  $R$  yields a picture already encountered in [24] for the BNW case and lower values of  $N$ .

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<sup>3</sup>As expected, type (b) programs give the same result as the program of (a) up to the orders 16, 14, 7; this fact can be used to validate the implementation of symmetries in these programs. In the BNW case, it is also possible to make a comparison with the  $N = 5$  expansion computed via Mathematica in [24]; again, there is agreement between the results of the Mathematica and Python programs.

<sup>4</sup>The results of the above numerical manipulations are reliable but, admittedly, not certified. Therefore, for the statements of the subsequent paragraphs stemming from this approach to the ODE (3.3) we have a sound computer indication of validity, rather than a “computer assisted proof”. Perhaps, certified results could be obtained implementing the calculation of  $\mathcal{D}_3, \mathcal{D}_4, \epsilon_3$ , a finite-difference scheme for the ODE in (3.3) and the related error analysis via any software for interval arithmetics.

Recalling that  $[0, T_c)$  is the domain of the maximal solution  $\mathcal{R}_3$  of (3.3), we can summarize this picture in the following way:

- (i) There is a critical number  $R_{crit}$ , depending on  $u_*$  and  $N$ , such that  $T_c = +\infty$  for  $0 \leq R \leq R_{crit}$ , and  $T_c < +\infty$  for  $R > R_{crit}$ . Moreover, for  $0 \leq R \leq R_{crit}$  one has  $\mathcal{R}_3(t) \rightarrow 0^+$  for  $t \rightarrow +\infty$ , while for  $R > R_{crit}$  one has  $\mathcal{R}_3(t) \rightarrow +\infty$  for  $t \rightarrow T_c^-$ .
- (ii) Let  $u$  denote the maximal solution of the NS Cauchy problem (1.21). Due to (i), for  $0 \leq R \leq R_{crit}$   $u$  is global and  $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$  for all  $t \in [0, +\infty)$ ; for  $R > R_{crit}$  it is only granted that  $[0, T_c)$  is in the domain of  $u$ , and that  $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$  for all  $t \in [0, T_c)$ . (Let us also recall Eq. (2.21), that can be used to bind the Fourier coefficients  $u(t) - u^N(t)$  via  $\mathcal{R}_3(t)$ .)
- (iii) For the data and the values of  $N$  considered in our computations (i.e., for  $N$  up to a maximum 20 or 12, depending on  $u_*$ ),  $R_{crit}$  increases with  $N$ .

One can associate to any  $R$  a physical Reynolds number  $Re$ ; of course, item (ii) implies global existence for the NS Cauchy problem when  $Re$  is below the critical value  $Re_{crit}$ , defined as in (1.24) with  $R$  replaced by  $R_{crit}$ . For the largest values of  $N$  attained in our computations for the BNW, TG and KM data,  $Re_{crit}$  is close to the values anticipated in the Introduction, i.e., 7.84, 5.07 and 1.00, respectively. In the sequel we give more specific information analyzing separately each one of the three initial data.

**The BNW datum.** This is

$$u_*(x_1, x_2, x_3) := 2(\cos(x_1 + x_2) + \cos(x_1 + x_3), \quad (3.5)$$

$$-\cos(x_1 + x_2) + \cos(x_2 + x_3), -\cos(x_1 + x_3) - \cos(x_2 + x_3)) ;$$

equivalently,

$$u_* = \sum_{a=1}^3 z_a (e_{k_a} + e_{-k_a}) , \quad (3.6)$$

$$k_1 := (1, 1, 0), \quad k_2 := (1, 0, 1), \quad k_3 := (0, 1, 1) ;$$

$$z_1 := (1, -1, 0) , \quad z_2 := (1, 0, -1) , \quad z_3 := (0, 1, -1) .$$

According to a conjecture of Behr, Nečas and Wu [2], this datum might produce a finite time blowup for the Euler equations (i.e., for NS in the limit case of zero viscosity); our position on this conjecture is described in [15].

Eqs. (1.22) (1.23) (1.24) for this datum give  $V_* = 2\sqrt{3}$ ,  $L_* = 2\pi/\sqrt{2}$  and

$$Re = 2\sqrt{6} \pi R = 15.39... R . \quad (3.7)$$



The symmetries of  $u_*$  were already discussed in [15], and are reviewed in Appendix A; in particular, the group  $\mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)$  has 12 elements. We already mentioned the investigation of [24] on the Reynolds expansion for the BNW datum, performed up to the order  $N = 5$  using Mathematica <sup>(5)</sup>; the conclusion of this analysis was a picture as in items (i)-(iii) before Eq.(3.5) where, for  $N = 5$ ,  $R_{crit} \in (0.23, 0.24)$  and, consequently,  $Re_{crit} \in (3.53, 3.70)$ .

As already indicated, our Python program implementing the BNW symmetries has allowed us to push the expansion up to the order  $N = 20$ . To give an idea of the computational complexity we mention that  $u_{20}$  has 6966 nonzero Fourier coefficients, whose wave vectors are partitioned in 638 orbits under the action of  $\mathcal{H}_r^+(u_*) \cup \mathcal{H}_r^-(u_*)$  on  $\mathbf{Z}^3$ . Moreover, the nonzero Fourier coefficients of  $u_{20}$  have very long expressions. For example, let us consider  $u_{20,k}^{(1)}$  for  $k = (1, 1, 0)$ , where <sup>(1)</sup> stands for the first of the three components:  $u_{20,(1,1,0)}^{(1)}(t)$  is a polynomial of degrees 9 in  $t$  and 386 in  $e^{-t}$  with very complicated rational coefficients.

Hereafter we summarize the results of the expansion up to  $N = 20$  and of its a posteriori analysis via (2.12). We have a picture as in the previously cited items (i)-(iii) where, for  $N = 20$ ,

$$R_{crit} \in (0.51, 0.52) \quad \text{whence} \quad Re_{crit} \in (7.84, 8.01) . \quad (3.8)$$

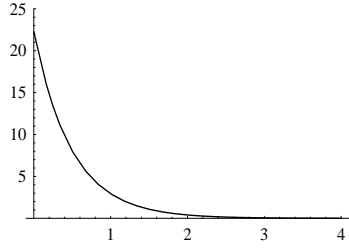
The forthcoming Boxes 1a-1d present some results about computations with  $N = 20$  and  $R = 0.51$ , giving information on the following functions of time: the quantity  $(2\pi)^{3/2}|u_k^{20}(t)|$  for the wave vector  $k = (1, 1, 0)$ ; the estimators  $\mathcal{D}_3$  and  $\epsilon_3$ ; the solution  $\mathcal{R}_3$  of the control Cauchy problem, which is global. In Boxes 2a-2d we consider the analogous functions in the case  $N = 20$  and  $R = 0.52$ , in which  $\mathcal{R}_3$  diverges at  $T_c = 2.855\dots$  <sup>(6)</sup>. Each one of these boxes (and of the subsequent ones) contains the graph of the function under consideration, and its numerical values for some choices of  $t$ .

Let us add a comment similar to one of [24] about the pictures that illustrated therein the  $N = 5$  BNW expansion. The functions in boxes of the types (a) and (b) (i.e.,  $(2\pi)^{3/2}|u_k^{20}(t)|$  and  $\mathcal{D}_3(t)$ ) are very similar in the cases  $R = 0.51$  and  $R = 0.52$ , even from the quantitative viewpoint. Boxes (c) indicate that, as for  $\epsilon_3$ , the difference between the cases  $R = 0.51$  and  $R = 0.52$  is quantitatively significant; this is sufficient to produce the completely different results for  $\mathcal{R}_3$  illustrated by boxes (d). Similar comments could be written about the boxes in the forthcoming paragraphs, illustrating our computations about the TG and KM data.

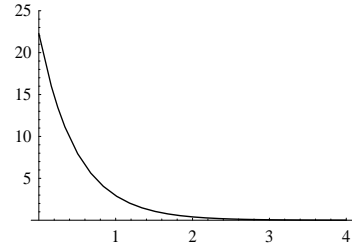
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<sup>5</sup>The a posteriori analysis of [24] was based on the tautological error estimators  $\mathcal{D}_m := \|u^5\|_m$  ( $m = 3, 4$ ) and  $\epsilon_3 := \|du^5/dt - \Delta u^5 - R\mathcal{P}(u^5, u^5)\|_3$ , that could be computed since the order  $N = 5$  is not too large. We repeat that, for the higher order computations in the present work, we have always used the rougher, but more easily computable estimators in (2.19) with  $M = 5$ .

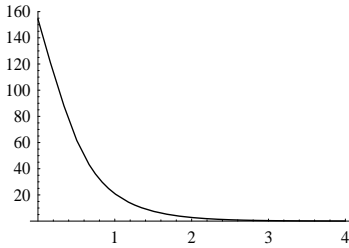
<sup>6</sup>An expression like  $r = a.bcd\dots$  means that  $a.bcd\dots$  are the first digits of the output in the numerical computation of  $r$ .



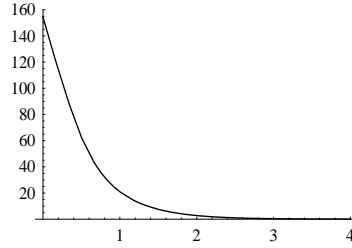
**Box 1a.** BNW,  $N = 20$ ,  $R = 0.51$ : the function  $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^{20}(t)|$ . One has  $\gamma(0) = 22.27\dots$ ,  $\gamma(0.5) = 8.031\dots$ ,  $\gamma(1) = 2.933\dots$ ,  $\gamma(1.5) = 1.077\dots$ ,  $\gamma(2) = 0.396\dots$ ,  $\gamma(4) = 7.261\dots \times 10^{-3}$ ,  $\gamma(8) = 2.435\dots \times 10^{-6}$ ,  $\gamma(10) = 4.461\dots \times 10^{-8}$ .



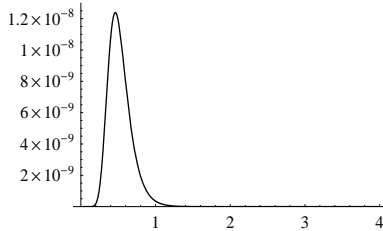
**Box 2a.** BNW,  $N = 20$ ,  $R = 0.52$ : the function  $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^{20}(t)|$ . One has  $\gamma(0) = 22.27\dots$ ,  $\gamma(0.5) = 8.025\dots$ ,  $\gamma(1) = 2.930\dots$ ,  $\gamma(1.5) = 1.076\dots$ ,  $\gamma(2) = 0.3960\dots$ ,  $\gamma(4) = 7.253\dots \times 10^{-3}$ .



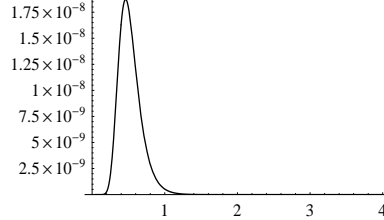
**Box 1b.** BNW,  $N = 20$ ,  $R = 0.51$ : the function  $\mathcal{D}_3(t)$ . One has  $\mathcal{D}_3(0) = 154.3\dots$ ,  $\mathcal{D}_3(0.5) = 62.32\dots$ ,  $\mathcal{D}_3(1) = 20.95\dots$ ,  $\mathcal{D}_3(1.5) = 7.505\dots$ ,  $\mathcal{D}_3(2) = 2.748\dots$ ,  $\mathcal{D}_3(4) = 0.05030\dots$ ,  $\mathcal{D}_3(8) = 1.687\dots \times 10^{-5}$ ,  $\mathcal{D}_3(10) = 3.091\dots \times 10^{-7}$ .



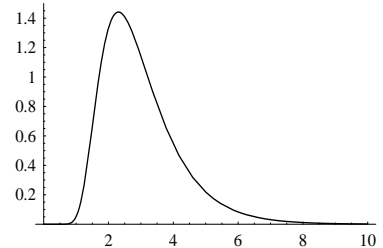
**Box 2b.** BNW,  $N = 20$ ,  $R = 0.52$ : the function  $\mathcal{D}_3(t)$ . One has  $\mathcal{D}_3(0) = 154.3\dots$ ,  $\mathcal{D}_3(0.5) = 62.53\dots$ ,  $\mathcal{D}_3(1) = 20.95\dots$ ,  $\mathcal{D}_3(1.5) = 7.498\dots$ ,  $\mathcal{D}_3(2) = 2.745\dots$ ,  $\mathcal{D}_3(4) = 0.05025\dots$ .



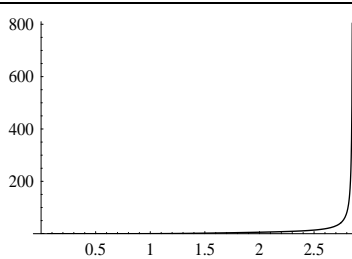
**Box 1c.** BNW,  $N = 20$ ,  $R = 0.51$ : the function  $\epsilon_3(t)$ . One has  $\epsilon_3(0) = 0$ ,  $\epsilon_3(0.46) = 1.239 \times 10^{-8}\dots$ ,  $\epsilon_3(0.8) = 1.895 \times 10^{-9}\dots$ ,  $\epsilon_3(1) = 3.453\dots \times 10^{-10}$ ,  $\epsilon_3(2) = 5.753\dots \times 10^{-14}$ ,  $\epsilon_3(4) = 5.695\dots \times 10^{-18}$ .



**Box 2c.** BNW,  $N = 20$ ,  $R = 0.52$ : the function  $\epsilon_3(t)$ . One has  $\epsilon_3(0) = 0$ ,  $\epsilon_3(0.46) = 1.868\dots \times 10^{-8}$ ,  $\epsilon_3(0.8) = 2.865\dots \times 10^{-9}$ ,  $\epsilon_3(1) = 5.219\dots \times 10^{-10}$ ,  $\epsilon_3(2) = 8.686\dots \times 10^{-14}$ ,  $\epsilon_3(4) = 8.572\dots \times 10^{-18}$ .



**Box 1d.** BNW,  $N = 20$ ,  $R = 0.51$ : the function  $\mathcal{R}_3(t)$ . This appears to be globally defined, and vanishing at  $+\infty$ . One has  $\mathcal{R}_3(0) = 0$ ,  $\mathcal{R}_3(1) = 0.05127\dots$ ,  $\mathcal{R}_3(1.5) = 0.6631\dots$ ,  $\mathcal{R}_3(2.3) = 1.441\dots$ ,  $\mathcal{R}_3(4) = 0.5433\dots$ ,  $\mathcal{R}_3(8) = 0.01143\dots$ ,  $\mathcal{R}_3(10) = 1.551\dots \times 10^{-3}$ .



**Box 2d.** BNW,  $N = 20$ ,  $R = 0.52$ : the function  $\mathcal{R}_3(t)$ . This diverges for  $t \rightarrow T_c = 2.855\dots$ . One has  $\mathcal{R}_3(0) = 0$ ,  $\mathcal{R}_3(0.5) = 1.847\dots \times 10^{-5}$ ,  $\mathcal{R}_3(1) = 0.1373\dots$ ,  $\mathcal{R}_3(1.5) = 2.013\dots$ ,  $\mathcal{R}_3(2) = 5.611\dots$ ,  $\mathcal{R}_3(2.85) = 804.5$ .

**The TG datum.** This is

$$u_*(x_1, x_2, x_3) := (\sin x_1 \cos x_2 \cos x_3, -\cos x_1 \sin x_2 \cos x_3, 0) ; \quad (3.9)$$

equivalently,

$$u_* = \frac{i}{8} \sum_{a=1}^4 z_a (e_{k_a} - e_{-k_a}) , \quad (3.10)$$

$$\begin{aligned} k_1 &:= (1, 1, 1), \quad k_2 := (1, 1, -1), \quad k_3 := (1, -1, 1), \quad k_4 := (-1, 1, 1) ; \\ z_1 &:= z_2 := (-1, 1, 0), \quad z_3 := (-1, -1, 0), \quad z_4 := -z_3 . \end{aligned}$$

The third component of  $u_*$  vanishes; however, this component does not vanish in the exact NS solution  $u$  with this datum <sup>(7)</sup>, nor in the coefficients  $u_1, u_2, \dots$  of the Reynolds expansion.

The above datum was considered by Taylor and Green in [30] for a pioneering computation of the dissipation rate of the kinetic energy via a Taylor expansion in time of the NS solution  $u$ . The same datum has been the subject of many subsequent investigations; among them we cite, in particular, [3]. These investigations treated sophisticated issues, such as the numerical verification of Kolmogorov's hypothesis on turbulence for very large  $R$ ; as already mentioned, global existence was essential assumed without proof for these large values of  $R$ .

Eqs. (1.22) (1.23) (1.24) for this datum give  $V_* = 1/2$ ,  $L_* = 2\pi/\sqrt{3}$  and

$$Re = \frac{\pi}{\sqrt{3}} R = 1.813... R . \quad (3.11)$$

The TG symmetries are described in Appendix A; in particular,  $\mathcal{H}_r^+(u_*)$  has 16 elements and coincides with  $\mathcal{H}_r^-(u_*)$ . As anticipated, we have used symmetry considerations to perform the Reynolds expansion up to the order  $N = 20$ .

Again for an appreciation of the computational complexity, we mention that  $u_{20}$  has 10560 nonzero Fourier coefficients, whose wave vectors are partitioned in 715 orbits under the action of  $\mathcal{H}_r^+(u_*)$  on  $\mathbf{Z}^3$ . As an example consider  $u_{20,k}^{(1)}(t)$  for  $k = (1, 1, 1)$ , where <sup>(1)</sup> denotes the first of the three components; this is a polynomial of degrees 9 in  $t$  and 547 in  $e^{-t}$ .

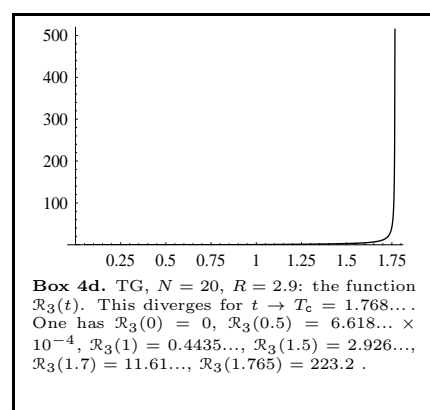
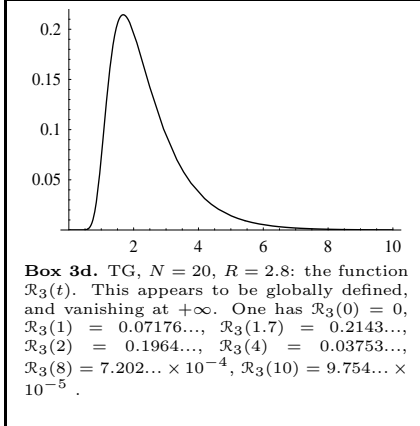
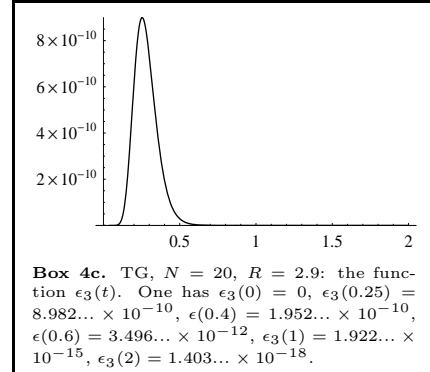
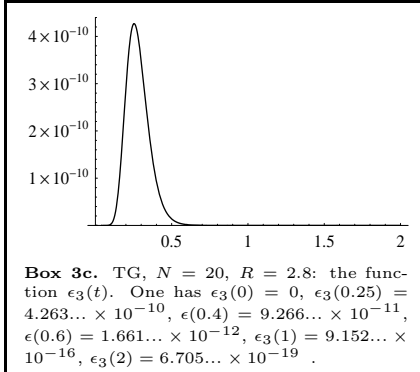
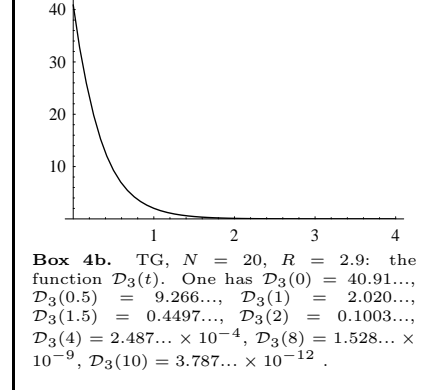
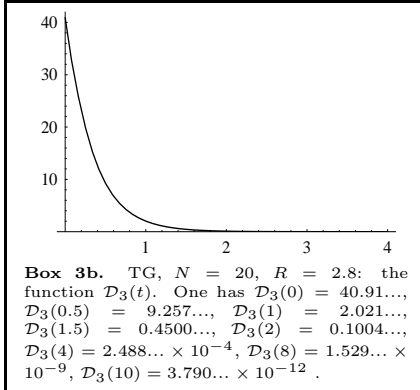
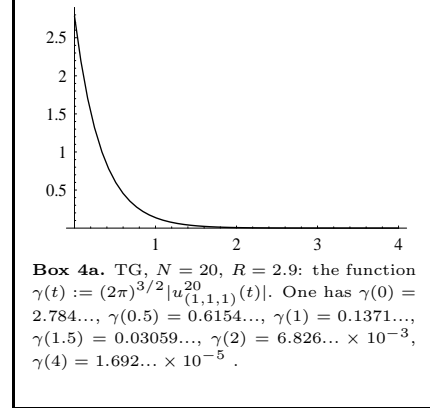
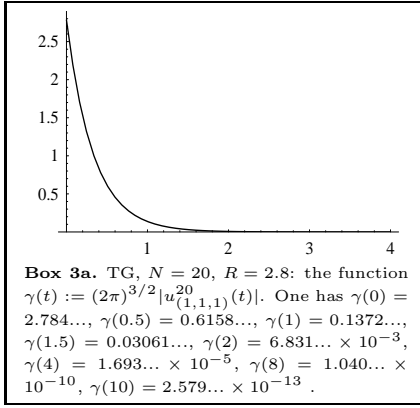
The expansion up to  $N = 20$  and its a posteriori analysis give a picture as in items (i)-(iii) before Eq.(3.5); for  $N = 20$  one has

$$R_{crit} \in (2.8, 2.9), \quad \text{whence} \quad Re_{crit} \in (5.07, 5.27) . \quad (3.12)$$

The forthcoming Boxes 3a-3d and 4a-4d present some results of these computations.

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<sup>7</sup>In fact, denoting with <sup>(3)</sup> the third component we have  $(du^{(3)}/dt)(0) = R\mathcal{P}(u_*, u_*)^{(3)}$ , which is nonzero if  $R \neq 0$



**The KM datum.** This is

$$u_*(x_1, x_2, x_3) := 2 \left( \sin x_1 \cos x_2 \cos x_3 (\cos 2x_2 - \cos 2x_3), \quad (3.13) \right.$$

$$\left. \cos x_1 \sin x_2 \cos x_3 (\cos 2x_3 - \cos 2x_1), \cos x_1 \cos x_2 \sin x_3 (\cos 2x_1 - \cos 2x_2) \right) .$$

Equivalently,

$$u_* = \frac{i}{8} \sum_{a=1}^{12} z_a (e_{k_a} - e_{-k_a}) , \quad (3.14)$$

$$\begin{aligned} k_1 &:= (3, 1, 1), & k_2 &:= (3, 1, -1), & k_3 &:= (1, 3, 1), & k_4 &:= (1, 3, -1), \\ k_5 &:= (1, 1, 3), & k_6 &:= (1, 1, -3), & k_7 &:= (1, -1, 3), & k_8 &:= (1, -1, -3), \\ k_9 &:= (1, -3, 1), & k_{10} &:= (1, -3, -1), & k_{11} &:= (3, -1, 1), & k_{12} &:= (3, -1, -1), \\ z_1 &:= (0, 1, -1), & z_2 &:= (0, 1, 1), & z_3 &:= z_9 := (-1, 0, 1), & z_4 &:= z_{10} := (-1, 0, -1), \\ z_5 &:= z_6 := (1, -1, 0), & z_7 &:= z_8 := (1, 1, 0), & z_{11} &:= -z_2, & z_{12} &:= -z_1 . \end{aligned}$$

This datum was the subject of an investigation started by Kida [10] and continued by Kida and Murakami [11]; it is maximally symmetric, in the sense that  $\mathcal{H}_r^+(u_*)$  is the full octahedral group  $O(3, \mathbf{Z})$ , and coincides with  $\mathcal{H}_r^-(u_*)$  (see Appendix A). In the cited works, this feature was used to reduce the computational costs in the solution of the NS equations via pseudo-spectral methods (with no discussion of the global existence problem, as typical of numerical investigations on turbulence).

Eqs. (1.22) (1.23) (1.24) for this datum give  $V_* = \sqrt{3}/2$ ,  $L_* = 2\pi/\sqrt{11}$  and

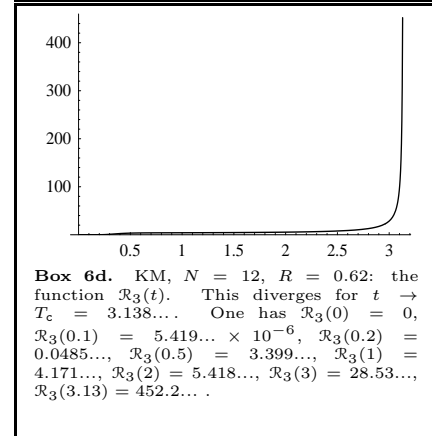
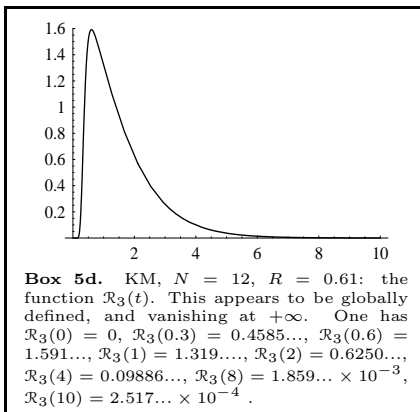
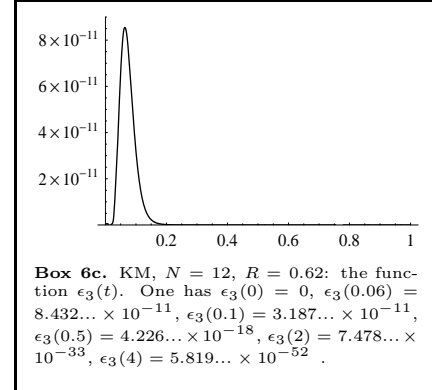
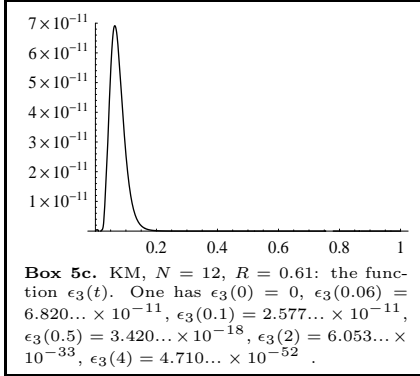
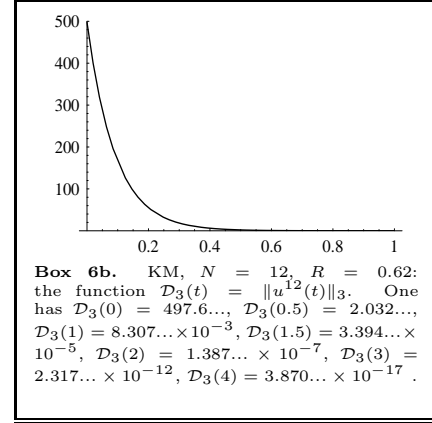
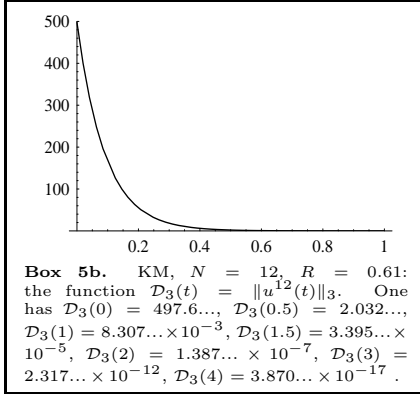
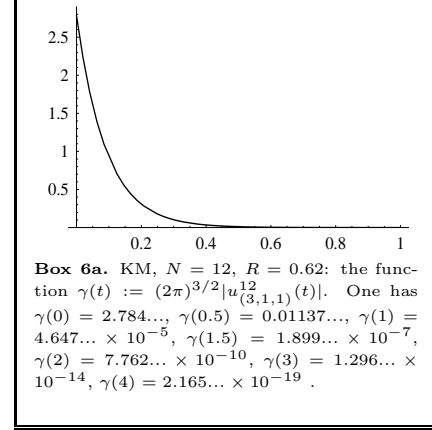
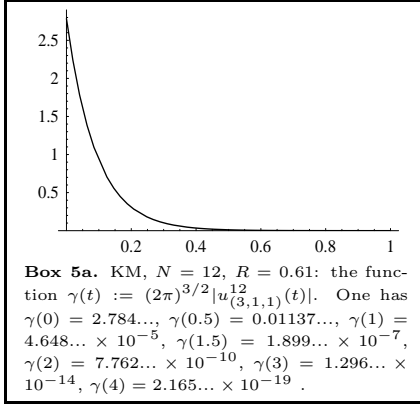
$$Re = \sqrt{\frac{3}{11}} \pi R = 1.640... R . \quad (3.15)$$

In the KM case, using the symmetries we could perform the Reynolds expansion up to the order  $N = 12$ . Let us mention that  $u_{12}$  has 33312 nonzero Fourier coefficients, whose wave vectors are partitioned in 797 orbits under the action of  $\mathcal{H}_r^+(u_*) = O(3, \mathbf{Z})$  on  $\mathbf{Z}^3$ . As an example consider  $u_{12,k}^{(2)}(t)$  for  $k = (3, 1, 1)$ , where  $^{(2)}$  denotes the second of the three components; this is a polynomial of degrees 5 in  $t$  and 867 in  $e^{-t}$ .

The result of computations up to  $N = 12$  is a picture as in items (i)-(iii) before (3.5); for  $N = 12$  one has

$$R_{crit} \in (0.61, 0.62), \quad \text{whence} \quad Re_{crit} \in (1.00, 1.02) . \quad (3.16)$$

The forthcoming Boxes 5a-5d and 6a-6d present some results of these computations. (Note that the numerical sample values reported in Boxes 5a and 6a, 5b and 6b are almost always the same to 4 meaningful digits; the situation is different for Boxes 5c and 6c, 5d and 6d.)



## 4 Concluding remarks

Let us propose a question that, in our opinion, is worthy of future consideration: keeping the general setting of the present paper, is it possible to improve the critical Reynolds numbers yielding global existence for the initial data considered here?

An obvious attempt one could make in this direction is to try higher order Reynolds expansions by means of more powerful computational utilities; however, it is not granted that this strategy would yield significant improvements <sup>(8)</sup>.

The problem of global existence for higher Reynolds numbers could be attacked via a different strategy. In this case the idea is to devise specific versions of the control Cauchy problem (2.12), fitted to the symmetries of the initial data under investigation. In particular, one could consider the basic inequality (1.15) and the Kato inequality (1.17) in the subspaces of  $\mathbb{H}_{\Sigma_0}^n$  and  $\mathbb{H}_{\Sigma_0}^{n+1}$  formed by the vector fields which have the same symmetry group as the initial datum; the constants  $K_n, G_n$  for the inequalities (1.15) (1.17) in these subspaces could be significantly smaller, and the control problem (2.12) with these smaller constants would yield global existence for higher Reynolds numbers.

However, estimating the constants for (1.15) (1.17) in the presence of symmetries requires some effort: one must adapt the approach of [21] [23] to the case where a symmetry group is specified and, especially, one must make anew rather expensive numerical computations to evaluate  $K_n$  and  $G_n$  for specific values of  $n$  and for the given symmetries. We plan to treat the above issues in future works.

**Acknowledgments.** This work was partly supported by INdAM, INFN and by MIUR, PRIN 2010 Research Project “Geometric and analytic theory of Hamiltonian systems in finite and infinite dimensions”.

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<sup>8</sup>It might happen that, for large  $R$  or large times, there is an optimal order  $N$  giving the best approximation of the exact NS solution and that, for larger  $N$ , the norms of  $u^N(t)$  and of its differential error increase, finally yielding worse results in the application of the control Cauchy problem. One could expect this to happen for values of  $R$  or times  $t$  for which the series  $\sum_{j=0}^{+\infty} R^j u_j(t)$  is not convergent; as already mentioned in [24], the convergence of this series is known for small  $R$  or short times, but bounds on  $R$  or  $t$  yielding convergence are not presently known with sufficient precision.

## A Appendix. Symmetries of the BNW, TG and KM data

Finding the symmetries or pseudo-symmetries of any NS initial datum  $u_*$  amounts to determine all pairs  $(S, a) \in O(d, \mathbf{Z}) \ltimes \mathbf{T}^d$  such that  $\mathcal{E}_*(S, a)u_* = \pm u_*$ . When  $u_*$  is a Fourier polynomial it is generally convenient to rephrase this equation in terms of Fourier coefficients via (2.27); the solutions can be obtained by automatic computations, say with Mathematica. These remarks apply, in particular, to the three initial data considered in this paper.

Throughout this Appendix we work in dimension  $d = 3$ , using the following notations:

$$\begin{aligned} D_1 &:= \text{diag}(1, 1, 1), & D_2 &:= \text{diag}(-1, 1, 1), & D_3 &:= \text{diag}(1, -1, 1), & D_4 &:= \text{diag}(1, 1, -1), \\ D_5 &:= -D_2, & D_6 &:= -D_3, & D_7 &:= -D_4, & D_8 &:= -D_1 \end{aligned}$$

$$\begin{aligned} Q_1 &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & Q_2 &:= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & Q_3 &:= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ Q_4 &:= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & Q_5 &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & Q_6 &:= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

Any matrix  $Q_\beta$  ( $\beta = 1, \dots, 6$ ) acts on elements of  $\mathbf{R}^3$  or  $\mathbf{T}^3$  applying to their components one of the 6 permutations of  $\{1, 2, 3\}$ . According to Eq. (2.23), the octahedral group  $O(3, \mathbf{Z})$  is formed by all matrices of the form

$$S_{\alpha\beta} := D_\alpha Q_\beta \quad (\alpha = 1, \dots, 8; \beta = 1, \dots, 6). \quad (\text{A.2})$$

To go on, we put

$$a_1 := (0, 0, 0), \quad a_2 := (\pi, 0, 0), \quad a_3 := (0, \pi, 0), \quad a_4 := (0, 0, \pi), \quad (\text{A.3})$$

$$a_5 := (\pi, \pi, 0), \quad a_6 := (\pi, 0, \pi), \quad a_7 := (0, \pi, \pi), \quad a_8 := (\pi, \pi, \pi)$$

(where  $\pi$  is an abbreviation for  $\pi \bmod 2\pi\mathbf{Z}$ ). With the above notations, the symmetries and pseudosymmetries of the BNW, TG and KM data can be described as follows.



**BNW case.** Let  $u_*$  denote the BNW datum (3.5). Then

$$\mathcal{H}^+(u_*) = \{(S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}, \quad (\text{A.4})$$

$$I := \{(1, 1, 1), (1, 1, 8), (1, 2, 4), (1, 2, 5), (1, 3, 2), (1, 3, 7),$$

$$(8, 4, 4), (8, 4, 5), (8, 5, 2), (8, 5, 7), (8, 6, 1), (8, 6, 8)\}.$$

This group has 12 elements; it was already described (with different notations) in [15], where it was shown to be isomorphic to the dihedral group  $\mathbf{D}_6$  (the group of orthogonal symmetries of a regular hexagon). The pseudo-symmetry space of the BNW datum contains  $(S_{81}, a_1) = (\text{diag}(-1, -1, -1), (0, 0, 0))$ ; thus

$$\mathcal{H}^-(u_*) = \{(S_{\alpha\beta}, a_\gamma) \circ (S_{81}, a_1) \mid (\alpha, \beta, \gamma) \in I\} = \{(-S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}. \quad (\text{A.5})$$

The reduced symmetry group and pseudo-symmetry space for this datum are

$$\mathcal{H}_r^\pm(u_*) = \{\pm S_{\alpha\beta} \mid (\alpha, \beta, \gamma) \in I\}; \quad (\text{A.6})$$

the above two sets are disjoint, and each one of them has 6 elements. In [15], it was shown that  $\mathcal{H}_r^+(u_*)$  is isomorphic to the dihedral group  $\mathbf{D}_3$  (the group of orthogonal symmetries of an equilateral triangle).

**TG case.** Let  $u_*$  denote the TG datum (3.9). Then

$$\mathcal{H}^+(u_*) = \{(S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}, \quad (\text{A.7})$$

$$I := \{(1, 1, 1), (1, 1, 5), (1, 1, 6), (1, 1, 7), (1, 4, 2), (1, 4, 3), (1, 4, 4), (1, 4, 8),$$

$$(2, 1, 1), (2, 1, 5), (2, 1, 6), (2, 1, 7), (2, 4, 2), (2, 4, 3), (2, 4, 4), (2, 4, 8),$$

$$(3, 1, 1), (3, 1, 5), (3, 1, 6), (3, 1, 7), (3, 4, 2), (3, 4, 3), (3, 4, 4), (3, 4, 8),$$

$$(4, 1, 1), (4, 1, 5), (4, 1, 6), (4, 1, 7), (4, 4, 2), (4, 4, 3), (4, 4, 4), (4, 4, 8),$$

$$(5, 1, 1), (5, 1, 5), (5, 1, 6), (5, 1, 7), (5, 4, 2), (5, 4, 3), (5, 4, 4), (5, 4, 8),$$

$$(6, 1, 1), (6, 1, 5), (6, 1, 6), (6, 1, 7), (6, 4, 2), (6, 4, 3), (6, 4, 4), (6, 4, 8),$$

$$(7, 1, 1), (7, 1, 5), (7, 1, 6), (7, 1, 7), (7, 4, 2), (7, 4, 3), (7, 4, 4), (7, 4, 8),$$

$$(8, 1, 1), (8, 1, 5), (8, 1, 6), (8, 1, 7), (8, 4, 2), (8, 4, 3), (8, 4, 4), (8, 4, 8)\}.$$

This group has 64 elements. The pseudo-symmetry space of the TG datum contains  $(S_{11}, a_8) = (\text{diag}(1, 1, 1), (\pi, \pi, \pi))$ ; thus

$$\mathcal{H}^-(u_*) = \{(S_{11}, a_8) \circ (S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\} = \{(S_{\alpha\beta}, a_8 + a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}. \quad (\text{A.8})$$

The reduced symmetry group and pseudo-symmetry space for this datum coincide; they have 16 elements, and are given by

$$\mathcal{H}_r^\pm(u_*) = \{S_{\alpha\beta} \mid (\alpha, \beta, \gamma) \in I\}. \quad (\text{A.9})$$

**KM case.** Let  $u_*$  denote the KM datum (3.13). Then

$$\mathcal{H}^+(u_*) = \{(S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}, \quad (\text{A.10})$$

$$I := \{ \begin{array}{llllllll} (1, 1, 1), & (1, 1, 5), & (1, 1, 6), & (1, 1, 7), & (1, 2, 1), & (1, 2, 5), & (1, 2, 6), & (1, 2, 7), \\ (1, 3, 1), & (1, 3, 5), & (1, 3, 6), & (1, 3, 7), & (1, 4, 2), & (1, 4, 3), & (1, 4, 4), & (1, 4, 8), \\ (1, 5, 2), & (1, 5, 3), & (1, 5, 4), & (1, 5, 8), & (1, 6, 2), & (1, 6, 3), & (1, 6, 4), & (1, 6, 8), \\ (2, 1, 1), & (2, 1, 5), & (2, 1, 6), & (2, 1, 7), & (2, 2, 1), & (2, 2, 5), & (2, 2, 6), & (2, 2, 7), \\ (2, 3, 1), & (2, 3, 5), & (2, 3, 6), & (2, 3, 7), & (2, 4, 2), & (2, 4, 3), & (2, 4, 4), & (2, 4, 8), \\ (2, 5, 2), & (2, 5, 3), & (2, 5, 4), & (2, 5, 8), & (2, 6, 2), & (2, 6, 3), & (2, 6, 4), & (2, 6, 8), \\ (3, 1, 1), & (3, 1, 5), & (3, 1, 6), & (3, 1, 7), & (3, 2, 1), & (3, 2, 5), & (3, 2, 6), & (3, 2, 7), \\ (3, 3, 1), & (3, 3, 5), & (3, 3, 6), & (3, 3, 7), & (3, 4, 2), & (3, 4, 3), & (3, 4, 4), & (3, 4, 8), \\ (3, 5, 2), & (3, 5, 3), & (3, 5, 4), & (3, 5, 8), & (3, 6, 2), & (3, 6, 3), & (3, 6, 4), & (3, 6, 8), \\ (4, 1, 1), & (4, 1, 5), & (4, 1, 6), & (4, 1, 7), & (4, 2, 1), & (4, 2, 5), & (4, 2, 6), & (4, 2, 7), \\ (4, 3, 1), & (4, 3, 5), & (4, 3, 6), & (4, 3, 7), & (4, 4, 2), & (4, 4, 3), & (4, 4, 4), & (4, 4, 8), \\ (4, 5, 2), & (4, 5, 3), & (4, 5, 4), & (4, 5, 8), & (4, 6, 2), & (4, 6, 3), & (4, 6, 4), & (4, 6, 8), \\ (5, 1, 1), & (5, 1, 5), & (5, 1, 6), & (5, 1, 7), & (5, 2, 1), & (5, 2, 5), & (5, 2, 6), & (5, 2, 7), \\ (5, 3, 1), & (5, 3, 5), & (5, 3, 6), & (5, 3, 7), & (5, 4, 2), & (5, 4, 3), & (5, 4, 4), & (5, 4, 8), \\ (5, 5, 2), & (5, 5, 3), & (5, 5, 4), & (5, 5, 8), & (5, 6, 2), & (5, 6, 3), & (5, 6, 4), & (5, 6, 8), \\ (6, 1, 1), & (6, 1, 5), & (6, 1, 6), & (6, 1, 7), & (6, 2, 1), & (6, 2, 5), & (6, 2, 6), & (6, 2, 7), \\ (6, 3, 1), & (6, 3, 5), & (6, 3, 6), & (6, 3, 7), & (6, 4, 2), & (6, 4, 3), & (6, 4, 4), & (6, 4, 8), \\ (6, 5, 2), & (6, 5, 3), & (6, 5, 4), & (6, 5, 8), & (6, 6, 2), & (6, 6, 3), & (6, 6, 4), & (6, 6, 8), \\ (7, 1, 1), & (7, 1, 5), & (7, 1, 6), & (7, 1, 7), & (7, 2, 1), & (7, 2, 5), & (7, 2, 6), & (7, 2, 7), \\ (7, 3, 1), & (7, 3, 5), & (7, 3, 6), & (7, 3, 7), & (7, 4, 2), & (7, 4, 3), & (7, 4, 4), & (7, 4, 8), \\ (7, 5, 2), & (7, 5, 3), & (7, 5, 4), & (7, 5, 8), & (7, 6, 2), & (7, 6, 3), & (7, 6, 4), & (7, 6, 8), \\ (8, 1, 1), & (8, 1, 5), & (8, 1, 6), & (8, 1, 7), & (8, 2, 1), & (8, 2, 5), & (8, 2, 6), & (8, 2, 7), \\ (8, 3, 1), & (8, 3, 5), & (8, 3, 6), & (8, 3, 7), & (8, 4, 2), & (8, 4, 3), & (8, 4, 4), & (8, 4, 8), \\ (8, 5, 2), & (8, 5, 3), & (8, 5, 4), & (8, 5, 8), & (8, 6, 2), & (8, 6, 3), & (8, 6, 4), & (8, 6, 8) \end{array} \}.$$

This group has 192 elements. As in the TG case, the pseudo-symmetry space of the KM datum contains  $(S_{11}, a_8) = (\text{diag}(1, 1, 1), (\pi, \pi, \pi))$ ; thus

$$\mathcal{H}^-(u_*) = \{(S_{11}, a_8) \circ (S_{\alpha\beta}, a_\gamma) \mid (\alpha, \beta, \gamma) \in I\} = \{(S_{\alpha\beta}, a_8 + a_\gamma) \mid (\alpha, \beta, \gamma) \in I\}. \quad (\text{A.11})$$

The reduced symmetry group and pseudo-symmetry space for the KM datum coincide; they have 48 elements, i.e., they coincide with the full octahedral group:

$$\mathcal{H}_r^\pm(u_*) = \{S_{\alpha\beta} \mid (\alpha, \beta) \in I\} = O(3, \mathbf{Z}). \quad (\text{A.12})$$

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